M303

Diagnostic quiz

Am I ready to start *Further pure mathematics* (M303)?

Further pure mathematics is a fascinating third level module, that builds on topics introduced in our second level module M208, *Pure mathematics*. It covers several topics that are central to much modern pure mathematics including rings and fields, and metric spaces. The better prepared you are for it the more time you will have to enjoy the mathematics, and the greater your chance of success. This diagnostic quiz is designed to help you decide whether you are ready to start studying M303, *Further pure mathematics* or whether some further preparation may help you.

The quiz contains questions, hints and solutions.

We suggest that you try each question before looking at the hint for that question. If you get stuck then look at the hint and then maybe try looking at a book (such as an M208 text book). You will not necessarily remember everything, and may well need to look up some things. This is perfectly all right. If you have studied the second level module M208, *Pure mathematics* (or its predecessor M203) then you will find all the necessary background material in this module's Handbook.

It may be that even after refreshing your memory of a particular topic, you are still unable to answer a particular question. In this case it is best not to dwell too long on the question and instead to look at the solutions at the end of the quiz.

If you struggle with a particular question, or set of questions, then you may find it helpful to revise the relevant sections of M208.

Do contact your Student Support Team via StudentHome if you have any queries about M303, or your readiness to study it.

Introductory material

This section covers general background material and is not related to any specific topics.

Set theory

Question 1

Which, if any, of the following are true?

(a) $1 \in \{1, 2, 3, 4, 5\}$,

(b) $\{0, 1\} \in \{0, 1, 2, 3, 4, 5\}$,

(c) $(0, 1) \in \{0, 1, 2, 3, 4, 5\}$,

(d) $\{0, 1\} \subseteq \{0, 1, 2, 3, 4, 5\}$.

Question 2

Which, if any, of the following are true?

(a) 1 is in the interval $[0, 1]$,

(b) 1 is in the interval $(0, 1)$,

(c) 1 is in the interval $(0, 1]$,

(d) 1 is in the interval $[0, 1)$.

Question 3

(a) Use set notation to specify the circle C of radius 4 centred at $(1, -3)$.

(b) Sketch the plane set $\{(x, y) \in \mathbb{R}^2 : x < 1\}$.

(c) Sketch the plane set $\{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 > 4\}$.

Set operations

Question 4

(a) Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Write the following as single sets.

(i) $A \cup B$,

(ii) $A \cap B$.

(b) Let $P = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ and $Q = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \leq 1\}$.

Sketch $P \cup Q$. (Use a broken line to indicate that the points on a boundary line are not included in a set and a solid line to indicate that they are included.)

Modular arithmetic and congruence

Question 5

(a) (i) Calculate $3 +_5 4$, giving your answer as an element of $\mathbb{Z}_5$.

(ii) Calculate $9 +_{11} 2$, giving your answer as an element of $\mathbb{Z}_{11}$.

(iii) Calculate $3 \times_{13} 12$, giving your answer as an element of $\mathbb{Z}_{13}$.

(b) Are the following true or false?

(i) $98 \equiv 29 \pmod{23}$,

(ii) $-4 \equiv 37 \pmod{19}$.

(c) Find all solutions, if any, of the following.

(i) $3 \times_7 x \equiv 5$,

(ii) $2 \times_8 x \equiv 4$,

(iii) $2 \times_4 x \equiv 3$.

Group theory

This material relates to the group theory sections of M208.

Groups

Question 6

Identify the identity element in each of the following group tables.

(a)

$$\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & a \end{array}$$

(b)

$$\begin{array}{c|cc} & p & q \\ \hline p & q & p \\ q & p & q \end{array}$$

(c)

$$\begin{array}{c|cccc} & t & s & w & x \\ \hline t & w & x & t & s \\ s & x & w & s & t \\ w & t & s & w & x \\ x & s & t & x & w \end{array}$$

Subgroups and cyclic groups

Question 7

(a) Consider the group G with the following Cayley table:

$\times_9$	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

(i) Write down the order of the group G.

(ii) Write down the order of each element of G.

(iii) Determine whether or not the group G is cyclic.

(iv) Find two proper subgroups of G.

Question 8

Find all the cyclic subgroups of $\mathbb{Z}_6$.

Analysis

This material relates to the analysis sections of M208.

Inequalities

Question 9

Prove the following inequalities:

(a) $\sqrt{a^2+b^2} \leq a+b$, for any non-negative real numbers a and b,

(b) $|a+b| \leq |a|+|b|$, for any real numbers a and b.

(c) Use the Triangle Inequality to prove the following statement.

$|a| \leq 3 \implies |3+a^3| \leq 30.$

Sequences

Question 10

(a) Write down the first 4 terms of the following sequences.

(i) The sequence with n^{th}-term given by

$$a_n = \frac{n}{n^2+1}.$$

(ii) The sequence with n^{th}-term given by

$$b_n = \frac{(-1)^n}{(n^2+2)}.$$

(iii) The sequence with n^{th}-term given by

$$c_n = 6.$$

(b) Determine whether the following sequences are monotonic.

(i) The sequence with n^{th}-term given by

$$a_n = n + 1.$$

(ii) The sequence with n^{th}-term given by

$$b_n = \frac{17}{n+2}.$$

(iii) The sequence with n^{th}-term given by

$$c_n = (-1)^n(n+1).$$

Question 11

(a) Use the fact that the sequence whose n^{th}-term is given by $\frac{1}{n}$ is null to deduce that the following are null, stating which rules you use.

(i) The sequence with n^{th}-term given by

$$a_n = \frac{1}{2n} + \frac{7}{n}.$$

(ii) The sequence with n^{th}-term given by

$$b_n = \frac{|\sin n|}{n}.$$

(b) For each of the following sequences either write down the limit or state that there is no limit.

(i) The sequence with n^{th}-term given by

$$a_n = \frac{3n+7}{n}.$$

(ii) The sequence with n^{th}-term given by

$$b_n = 5.$$

(iii) The sequence with n^{th}-term given by

$$c_n = (-1)^n \times 7.$$

Hints

This list contains a hint for each question. You should try the question before looking at the hint and try the question again, using the hint, before looking at the solution.

Set theory hints

Hint 1

Remember that $\in$ means 'is an element of' and $\subseteq$ means 'is a subset of'. Also remember that $\{x\}$ is a set and so is different to x.

Hint 2

Remember that the interval (a, b) with curved brackets contains all numbers x with $a < x < b$ but not the end points, a and b whereas the interval $[a, b]$ with square brackets contains all numbers x with $a \leq x \leq b$ so does include the end points. The interval $(a, b]$ contains all numbers x with $a < x \leq b$ and the interval $[a, b)$ contains all numbers x with $a \leq x < b$ so does include the end points.

Note that the notation used for the interval $(a, b) \subset \mathbb{R}$ is the same as the notation used for the point $(a, b) \in \mathbb{R}^2$. This can sometimes be confusing but it is usually clear from the context whether we are talking about an interval in the real line or a point in the plane.

Hint 3

Remember that the equation for a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

Set operation hints

Hint 4

Remember that $\cup$ means 'union' (the elements in either set) and $\cap$ means 'intersection' (elements common to both sets).

Modular arithmetic and congruence hints

Hint 5

For any integer $n \geq 2$,

$$\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}.$$

For a and b in $\mathbb{Z}_n$, the operations $\boldsymbol{+_n}$ and $\boldsymbol{\times_n}$ are defined by:

$a +_n b$ is the remainder of $a + b$ on division by n;

$a \times_n b$ is the remainder of $a \times b$ on division by n.

The integer n is called the **modulus** for this arithmetic.

Two integers a and b have a **common factor** c, where c is a natural number, if a and b are both divisible by c.

Two integers a and b are **coprime** (or **relatively prime**) if their only common factor is 1.

The **highest common factor** (**HCF**) of two integers a and b is their largest common factor.

Remember also that the linear equation $a \times_n x \equiv b$ has one unique solution if a and n are coprime. If, however, a and n are not coprime and have a highest common factor d, then $a \times_n x \equiv b$ has no solutions if d does not divide b and has d solutions if d does divide b.

Groups hints

Hint 6

Remember that a **group** is a set, G together with a binary operation $\circ$ defined on G, where the following rules hold.

G1 Closure For all g, h in G,

$$g \circ h \in G.$$

G2 Associativity For all g, h, k in G,

$$g \circ (h \circ k) = (g \circ h) \circ k.$$

G3 Identity There is an element e in G such that

$$g \circ e = g = e \circ g \quad \text{for all } g \text{ in } G.$$

(This element is an **identity element** for $\circ$ on G.)

G4 Inverses For each element g in G, there is an element h in G such that

$$g \circ h = e = h \circ g.$$

(The element h is an **inverse element** of g with respect to $\circ$.)

We often represent groups using **Cayley tables** like this one.

$\circ$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

This table tells us that $e \circ e = e$, $e \circ a = a$ and $b \circ a = e$.

Subgroups and cyclic groups hints

Hint 7

Remember that the **order** of a finite group G is the number of elements in G. If x is an element of a finite group $(G, \circ)$, with identity e, then the **order** of g is the smallest positive integer n such that $x^n = e$. If a group G or order n has an element of order n then G is a cyclic group.

Hint 8

Remember that $\mathbb{Z}_n$ is the group that has $\{0, 1, \ldots, n-1\}$ as a set and $+_n$ as the operation.

If x is an element of a group $(G, \circ)$ then the set of all 'powers' of x (that is $\{x, x \circ x, x \circ x \circ x, \ldots\}$ is called the subset of G **generated** by x. It is written as $\langle x \rangle$. So, for example, in $\mathbb{Z}_8$, the group with $\{0, 1, 2, 3, 4, 5, 6, 7\}$ as its set and $+_8$ as operation, the subgroup generated by 2, that is $\langle 2 \rangle$ is $\{2, 4, 6\}$ because $2 +_8 2 = 4$, $2 +_8 2 +_8 2 = 6$ and adding 2 again to 6 modulo 8 takes us back to 0 again.

If G is a group and there is an element $x \in G$ such that $G = \langle x \rangle$, then G is called a **cyclic group**.

Inequalities hints

Hint 9

Remember that for $a, b > 0$ we have $a > b \implies a^2 > b^2$ so try squaring both sides of the inequalities.

Remember that if $a, b \in \mathbb{R}$, then

(a) $|a + b| \leq |a| + |b|$ (usual form of **triangle inequality**,)

(b) $|a - b| \geq \big||a| - |b|\big|$ ('backwards' form of triangle inequality).

Sequences hints

Hint 10

Remember that a sequence with terms a_n is said to be

- **constant** if

  $$a_{n+1} = a_n, \quad \text{for } n = 1, 2, \ldots;$$

- **increasing** if

  $$a_{n+1} \geq a_n, \quad \text{for } n = 1, 2, \ldots;$$

  and **strictly increasing** if

  $$a_{n+1} > a_n, \quad \text{for } n = 1, 2, \ldots;$$

- **decreasing** if

  $$a_{n+1} \leq a_n, \quad \text{for } n = 1, 2, \ldots;$$

  and **strictly decreasing** if

  $$a_{n+1} < a_n, \quad \text{for } n = 1, 2, \ldots;$$

- a sequence is **monotonic** if it is either increasing or decreasing.

Hint 11

Remember that if two sequences with terms given by a_n and b_n are null, then:

Sum Rule the sequence with terms given by $a_n + b_n$ is null.

Multiple Rule The sequence with terms given by λa_n is null, for any real number λ.

Product Rule The sequence with terms given by $a_n b_n$ is null.

Squeeze Rule for null sequences
If the sequence with non-negative terms b_n is a null sequence, and

$$|a_n| \leq b_n, \quad \text{for } n = 1, 2, \ldots,$$

then the sequence with terms a_n is null.

The sequence with terms a_n is **convergent** with **limit** $\boldsymbol{l}$ if $(a_n - l)$ is a null sequence. (For example, the sequence with terms $a_n = 7 + \frac{1}{n}$ is convergent with limit 7 because the sequence given by $b_n = a_n - 7 = \frac{1}{n}$ is null).

Solutions to questions

Remember to try the hints before looking at the solutions.

Introductory material

Solution 1

(a) True

(b) False

(c) False

(d) True

Solution 2

(a) True

(b) False

(c) True

(d) False

Solution 3

(a) $C = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y + 3)^2 = 16\}$.

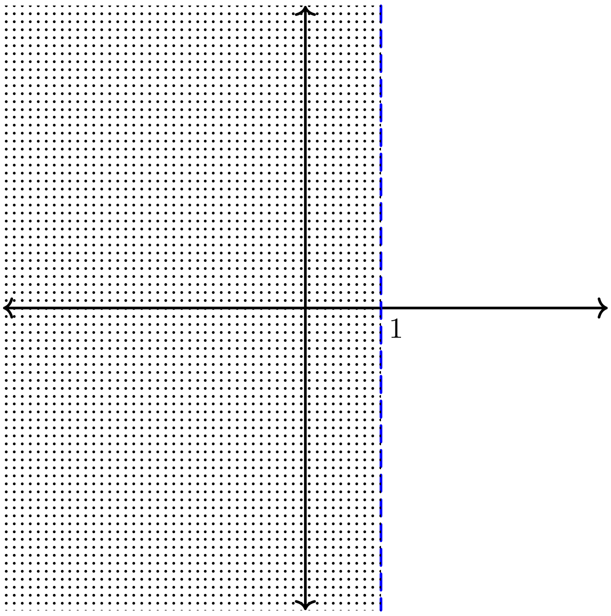
Note that there is nothing special about the letters x and y.

Other solutions such as

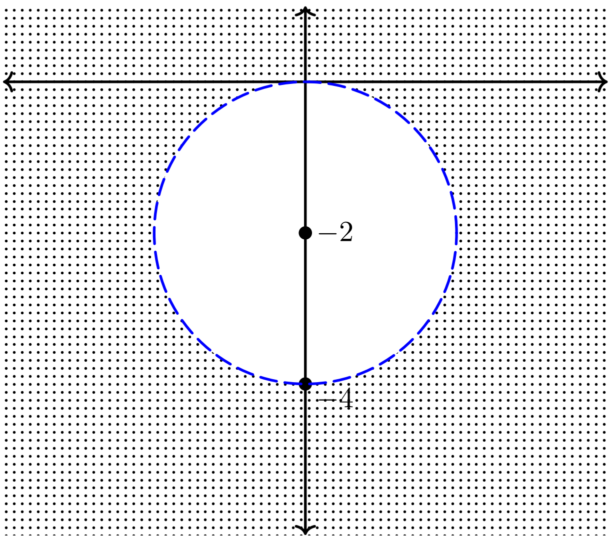
$C = \{(a, b) \in \mathbb{R}^2 : (a - 1)^2 + (b + 3)^2 = 16\}$ or

$C = \{(p, q) \in \mathbb{R}^2 : (p - 1)^2 + (q + 3)^2 = 16\}$ are also correct.

(b) This set is a vertical half-plane with the boundary line excluded, as follows.



(c) This set is the area outside of the circle with radius $2 = \sqrt{4}$ and with centre at $(0, -2)$. The set does not include the circle.



Solution 4

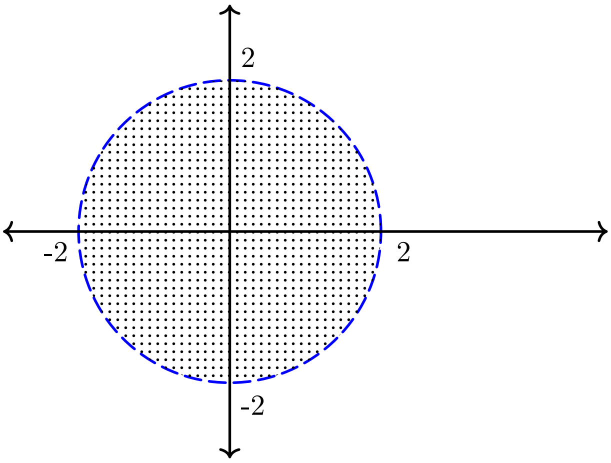
(a) (i) $\{1, 2, 3, 4, 5\}$

(ii) $\{3\}$

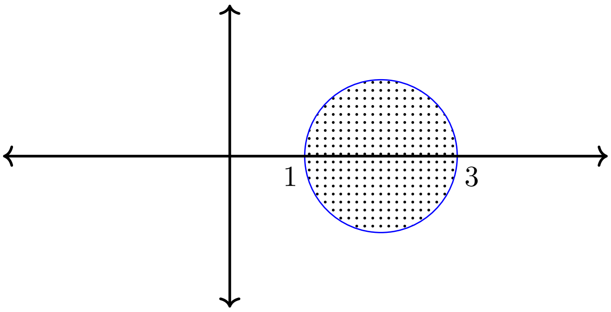
(b) This set is the area inside two overlapping circles, one with centre $(0, 0)$ and radius 2 and the other with centre $(0, 1)$ and radius 3. The boundary of the second, smaller circle is included.

Remember that the equation for a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

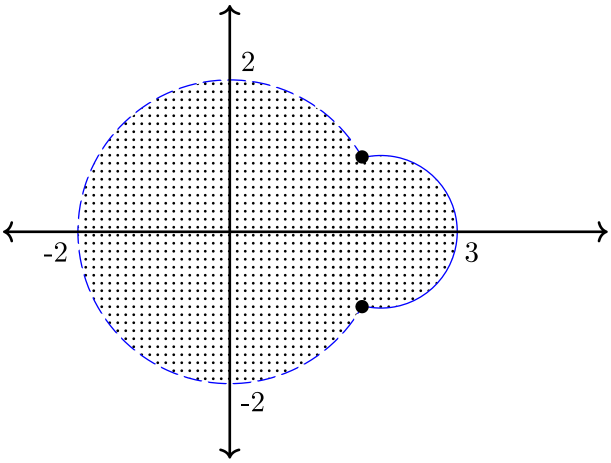
Sketch of P



Sketch of Q



Sketch of $P \cup Q$.



Note that we have used solid dots to indicate that the 'corner' points are included. (We adopt the convention that a solid dot, $\bullet$, indicates that a point is part of a set and an open dot, $\circ$, indicated that a point is not part of a set.)

Solution 5

(a) (i) $3 +_5 4 = 2$,

(ii) $9 +_{11} 2 = 0$,

(iii) $3 \times_{13} 12 = 10$.

(b) (i) We have that $98 \equiv 6 \pmod{23}$ and $29 \equiv 6 \pmod{23}$ so it is true that $98 \equiv 29 \pmod{23}$.

(ii) We have that $-4 \equiv 15 \pmod{19}$ and $37 \equiv 18 \pmod{19}$. Since 15 and 18 are not congruent $\pmod{19}$ (they are clearly different elements of $\mathbb{Z}_{19}$) it is false that $-4 \equiv 37 \pmod{19}$.

(c) (i) Considering $3 \times_7 x \equiv 5$. Noting that 3 and 7 are coprime, we expect a unique solution. We can write out all the multiples of $3 \pmod 7$

$$\begin{array}{ccccl} 0 & \times & 3 & \equiv & 0 \pmod 7 \\ 1 & \times & 3 & \equiv & 3 \pmod 7 \\ 2 & \times & 3 & \equiv & 6 \pmod 7 \\ 3 & \times & 3 & \equiv & 2 \pmod 7 \\ 4 & \times & 3 & \equiv & 5 \pmod 7 \\ 5 & \times & 3 & \equiv & 1 \pmod 7 \\ 6 & \times & 3 & \equiv & 4 \pmod 7 \end{array}$$

showing that 4 is the unique solution.

(ii) Considering $2 \times_8 x \equiv 4$. Noting that 2 is the highest common factor of 2 and 8 and that 2 divides 4, we expect 2 solutions. We can write out all the multiples of $2 \pmod 8$

$$\begin{array}{ccccl} 0 & \times & 2 & \equiv & 0 \pmod 8 \\ 1 & \times & 2 & \equiv & 2 \pmod 8 \\ 2 & \times & 2 & \equiv & 4 \pmod 8 \\ 3 & \times & 2 & \equiv & 6 \pmod 8 \\ 4 & \times & 2 & \equiv & 0 \pmod 8 \\ 5 & \times & 2 & \equiv & 2 \pmod 8 \\ 6 & \times & 2 & \equiv & 4 \pmod 8 \\ 7 & \times & 2 & \equiv & 6 \pmod 8 \end{array}$$

showing that 2 and 6 are the two solutions.

(iii) Considering $2 \times_4 x \equiv 3$. Noting that 2 is the highest common factor of 2 and 4 and that 2 does not divide 3, we expect no solutions. We can check by multiplying.

$$\begin{array}{ccccl} 0 & \times & 2 & \equiv & 0 \pmod 4 \\ 1 & \times & 2 & \equiv & 2 \pmod 4 \\ 2 & \times & 2 & \equiv & 0 \pmod 4 \\ 3 & \times & 2 & \equiv & 2 \pmod 4 \end{array}$$

Group theory

Solution 6

(a) The first row and the first column repeat the borders of the table, so the identity is a.

(b) The second row and the second column repeat the borders of the table, so the identity is q.

(c) The third row and the third column repeat the borders of the table, so the identity is w.

Solution 7

(a) There are 6 elements in the set for G, so the order of G is 6.

(b) The element 1 has order 1. The element 8 has order 2, the elements 4 and 7 have order 3, and the elements 2 and 5 have order 6.

(c) The group is cyclic because either of the elements of order 6 will generate it.

(d) The elements 1 and 8 form a subgroup of order 2, and the elements 1, 4 and 7 form a subgroup of order 3. There are no others since the trivial subgroup and G are not *proper* sibgroups.

Solution 8

Since 0 is the identity element in $(\mathbb{Z}_6, +_6)$, we have

$$\langle 0 \rangle = \{0\}.$$

To find the cyclic subgroup $\langle 1 \rangle$, we need to keep adding 1 to itself (working modulo 6) until 1 is reached again.

This gives

$$1, 2, 3, 4, 5, 0, 1, \ldots, .$$

Since 5 is the inverse of 1, it will generate the same subgroup so

$$\langle 1 \rangle = \{0, 1, 2, 3, 4, 5\},$$
$$\langle 5 \rangle = \{0, 1, 2, 3, 4, 5\}.$$

To find the cyclic subgroup $\langle 2 \rangle$, we need to keep adding 2 to itself (working modulo 6) until 2 is reached again. This gives

$$2, 4, 0, 2, \ldots.$$

and 4 is the inverse of 2, so

$$\langle 2 \rangle = \{0, 2, 4\},$$
$$\langle 4 \rangle = \{0, 2, 4\}.$$

Finally, 3 is self-inverse, so

$$\langle 3 \rangle = \{0, 3\}.$$

Analysis

Solution 9

(a) Here

$$\begin{aligned} & \sqrt{a^2+b^2} \le a+b, \\ \Leftrightarrow \quad & a^2+b^2 \le (a+b)^2, \text{ for non-negative } a \text{ and } b, \\ & \qquad\quad = a^2+2ab+b^2, \\ \Leftrightarrow \quad & \qquad\quad 0 \le 2ab, \text{ on rearranging.} \end{aligned}$$

Since a and b are both non-negative, it follows that $2ab \ge 0$. The above chain of equivalent statements shows that we must then also have $\sqrt{a^2+b^2} \le a+b$.

(b) Here

$$\begin{aligned}
&|a+b| &&\le |a|+|b|,\\
\Leftrightarrow\quad &|a+b|^2 &&\le (|a|+|b|)^2,\\
\Leftrightarrow\quad &(a+b)^2 &&\le (|a|+|b|)^2,\\
\Leftrightarrow\quad &a^2+2ab+b^2 &&\le a^2+2|a|\times|b|+b^2,\\
\Leftrightarrow\quad &2ab &&\le 2|a|\times|b|.
\end{aligned}$$

For any real numbers a and b, we have that $2ab \le 2|a| \times |b|$. The above chain of equivalent statements shows that we must then also have $|a+b| \le |a|+|b|$.

(c) Suppose that $|a| \le 3$. The Triangle Inequality gives

$$\begin{aligned}
|3+a^3| &\le |3|+|a^3|,\\
&= 3+|a|^3.
\end{aligned}$$

Now $|a| \le 3$ and therefore

$$3+|a|^3 \le 3+27 = 30.$$

Therefore

$$|a| \le 3 \implies |3+a^3| \le 30.$$

Solution 10

(a) (i) $a_1 = \frac{1}{1^2+1} = \frac{1}{2}, a_2 = \frac{2}{2^2+1} = \frac{2}{5}$, similarly, $a_3 = \frac{3}{10}, a_4 = \frac{4}{17}$.

(ii) $b_1 = \frac{-1}{3}, b_2 = \frac{1}{6}, b_3 = \frac{-1}{11}, b_4 = \frac{1}{18}$.

(iii) $c_1 = c_2 = c_3 = c_4 = 6$.

(b) (i) $a_n = n+1$ and $a_{n+1} = n+2$. Since $n+2 > n+1$, this sequence is (strictly) increasing and so is monotonic.

(ii) $b_n = \frac{17}{n+2}$ and $b_{n+1} = \frac{17}{n+3}$. Since $\frac{17}{n+3} < \frac{17}{n+2}$ this sequence is (strictly) decreasing and so is monotonic.

(iii) Writing down the first few terms of the sequence gives: $-2, +3, -4, +5, \ldots$. This shows that the sequence is neither increasing or decreasing and so is not monotonic.

Solution 11

(a) (i) $\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n}$ and $\frac{7}{n} = 7 \times \frac{1}{n}$ so this sequence is null by the multiple and sum rules for null sequences.

(ii) $0 \le \frac{|\sin n|}{n} \le \frac{1}{n}$ so this sequence is null by the Squeeze rule.

(b) (i) This sequence is convergent with limit 3. This is because we can rewrite $a_n = \frac{3n+7}{n}$ as $a_n = 3 + \frac{7}{n}$ and see that the sequence with the n^{th}-term given by $a_n - 3 = \frac{7}{n}$ is null (since it is a multiple of the basic null sequence with the n^{th}-term given by $\frac{1}{n}$).

(ii) this sequence is convergent with limit 5. This is because the sequence with the n^{th}-term given by $b_n - 5 = 0$ is null (since it is the constant sequence with all terms equal to zero).

(iii) Writing out a few terms of this sequence gives $-7, +7, -7, +7, \ldots$. It is an alternating sequence and has no limit (and so is divergent).